Practice problems

1. Independent RV's

   (a) Suppose $X \stackrel{d}{=} \mathrm{N}[0,1]$ and $Y \stackrel{d}{=} \mathrm{Exp}[5]$ are independent random variables. Find the joint density $f_{XY}(x,y)$.

   (b) Suppose $X \stackrel{d}{=} \mathrm{Poisson}[3]$ and $Y \stackrel{d}{=} \mathrm{Binom}[3,0.6]$ are independent random variables. Find their joint probability mass function.

2. Suppose $X \stackrel{d}{=} \mathrm{Unif}[0,2]$

   (a) Compute the moment generating function $\mathrm{E}[e^{tX}]$.

   (b) Compute the probability density $f_Y(y)$ for the random variable $Y = \log X$.

3. Suppose a dice is painted so that five of the sides are numbered "1" and the sixth side is numbered "10". Random variables $X_1, X_2, \ldots, X_{100}$ are the values of 100 rolls of this dice.

   (a) Compute the expected value $E[X_1]$ and variance $\mathrm{Var}[X_1]$.

   (b) Use the central limit theorem to give an approximate probability density for the sum $X_1 + X_2 + \ldots + X_{100}$.

4. By the weak law of large numbers, the sample mean of many dice rolls converges in probability to the expected value of a dice roll (= 3.5). Use Chebyshev's inequality to bound how many times the dice must be rolled for a 90% chance that the sample mean is between 3.4 and 3.6.

5. Suppose random variables X, Y have joint density given by

$$f_{XY}(x,y) = \begin{cases} C(xy^2 + x^2y) & \text{if } 0 < x < y < 1 \\ 0 & \text{else} \end{cases}$$

   (a) Compute the constant C.

   (b) Compute the conditional probability $P\{X < \frac{1}{2} | Y < \frac{1}{2}\}$

   (c) Compute the conditional density $f_{X|Y}\left(x|\frac{3}{4}\right)$

   (d) Compute the conditional probability $P\{X < \frac{1}{2} | Y = \frac{3}{4}\}$

   (e) Are X and Y independent random variables? How do you know?

6. Use the method of Laplace transform to solve the following differential equation for the function $y(t)$

$$\begin{aligned} y' + 6y =& \delta_1(t) \\ y(0) =& 2 \end{aligned}$$

7. Use the table

$f(x)$	$\widehat{f}(\xi)$
$\mathbb{1}_{[-a,a]}(x)$	$\dfrac{\sin(a\xi)}{\pi\xi}$
$f(x-c)$	$e^{-ic\xi}\widehat{f}(\xi)$
$f(cx)$	$\dfrac{1}{c}\widehat{f}\left(\dfrac{\xi}{c}\right)$

   to compute the Fourier transform of the function

$$f(x) = \begin{cases} 2 & \text{if } -3 < x < -1 \\ -1 & \text{if } 4 < x < 8 \\ 0 & \text{else} \end{cases}$$

8. Use the Cooley-Tukey algorithm to compute the discrete Fourier transform of the following sample

t	0	$\pi/2$	π	$3\pi/2$
$x(t)$	1.5	2.5	-0.5	-1.0

9. Compute the indefinite integral from 0 to x of the following function.

$$f(t) = 3\delta_{-2}(t) + \mathbb{1}_{[-5,5]}(t)$$

Check that the distributional derivitive of your answer is $f(x)$

10. Suppose $\widehat{f}(\xi) = \frac{1}{1+\xi^2}$ and $\widehat{g}(\xi) = e^{-|x|}$. Compute the Fourier transform of the convolution $f * g$.